GFD I, 2/6/2012

Problem Set #3 Solutions

1) [10+10 points] For the ocean case the governing equation is

$$u_t - fv = -g\eta_x \qquad (*)$$

In geostrophic balance this gives:

$$v_g = \frac{g\eta_x}{f} = \frac{\left(9.8 \text{ m s}^{-2}\right)\left(10^{-5}\right)}{10^{-4} \text{ s}^{-1}} \cong 1.0 \text{ m s}^{-1}$$

And note that this is a flow to the North for a pressure gradient that pushes to the West. Ignoring the Coriolis term, the time it would take to accelerate to this *speed* (I am not being picky about the *direction* here) is found from a time integral of (*) with $f = 0$. Thus:

$$\Delta t = \frac{\text{change in speed}}{g\eta_x} = \frac{1.0 \text{ m s}^{-1}}{\left(9.8 \text{ m s}^{-2}\right)\left(10^{-5}\right)} \cong 10^4 \text{ s} \approx 3 \text{ hours}$$

In this time the parcel would have traveled a distance given by the standard physics formula:

$$\text{distance traveled} = \frac{1}{2} \times \text{ acceleration } \times \Delta t^2 = 0.5 \times \left(10^{-4} \text{ m s}^{-2}\right) \times \left(10^4 \text{ s}\right)^2 = 5 \text{ km}$$

This is not very far! The point of this exercise is to make it clear that the Coriolis force becomes important rather quickly compared with a day, and can substantially limit the speed attained by fluid motion.

In the atmospheric case the governing equation is

$$v_t + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} = -g\left.\frac{\partial z}{\partial y}\right|_p = 3.2 \times 10^{-3} \text{ m s}^{-2}$$

Assuming $f = 10^{-4} \text{ s}^{-1}$ the eventual geostrophic velocity is

$$u_g = \frac{1}{f}\left(-g\left.\frac{\partial z}{\partial y}\right|_p\right) = 32 \text{ m s}^{-1}$$

Note that this is a velocity to the East (which would be called a "Westerly" by atmospheric scientists). This direction of zonal flow is typical of mid-latitude winds at altitude in both hemispheres. Proceeding in the same way as for the ocean case, it would take about

$$\Delta t = \frac{\text{change in speed}}{\text{pressure force per unit mass}} = \frac{1}{f} \approx 3 \text{ hours}$$

For a non-rotating flow to accelerate to this speed, and during this time a fluid parcel would have moved about

$$\text{distance traveled} = \frac{1}{2} \times \text{ acceleration } \times \Delta t^2 = 0.5 \times \left(3.2 \times 10^{-3} \text{ m s}^{-2}\right) \times \left(10^4 \text{ s}\right)^2 = 180 \text{ km}$$

So, comparing the two cases (an assuming the sizes of the pressure gradients we used to be typical of the two fluids), they both accelerate to a speed equal to the eventual geostrophic speed in just a few hours, but the atmospheric flow goes over 30 times farther in this time, and attains a speed which is over 30 times faster.

2) [20 points] Including the effects of planetary rotation, the first part of derivation of the Hydrostatic Approximation is identical to that done in class lecture 2.4. That is, from scaling of the MASS equation with small density perturbations, we still find

$$W = U\frac{H}{L}, \text{ and therefore } \left[\frac{D}{Dt}\right] = \frac{U}{L}.$$

Next we want to scale the X,Y-MOM equations, in order to find a dynamically consistent scale for the pressure perturbations. Doing this with rotation we find:

$$\left[\rho\frac{D\mathbf{u}_H}{Dt} + \rho f\,\hat{\mathbf{k}} \times \mathbf{u}_H = -\nabla_H p\right]$$

$$\underbrace{\rho_{00}\frac{U^2}{L}}_{(1)} + \underbrace{\rho_{00} fU}_{(2)} = \underbrace{\frac{[p']}{L}}_{(3)}$$

And so the ratio of the first two terms is

$$\frac{(1)}{(2)} = \frac{U}{fL} \equiv Ro\,, \quad \text{the "Rossby Number"}$$

We have already done the rest of the scaling in the limit of high Ro (where we would neglect rotation. So here let us assume that $Ro << 1$, in which case the scaling for the pressure perturbation is

$$\left[p'\right] = \rho_{00} fUL \qquad (+)$$

The final step in the derivation of the Hydrostatic Approximation was to scale the Z-MOM equation. Doing this with our new scale (+) for the perturbation pressure, we find

$$\left[\rho \frac{Dw}{Dt} = \cancel{-\bar{p}_z} - p'_z \cancel{-\bar{\rho} g} - \rho' g\right] \text{ gives}$$

$$\rho_{00}\frac{WU}{L} = \frac{\left[p'\right]}{H} - \left[\rho'\right]g \text{ , which may be written as:}$$

$$\underbrace{\rho_{00} U \frac{H}{L}\frac{U}{L}}_{(1)} = \underbrace{\frac{\rho_{00} fUL}{H}}_{(2)} - \underbrace{\left[\rho'\right]g}_{(3)}$$

The ratio of the first two terms is now given by

$$\frac{(1)}{(2)} = \frac{\text{vertical acceleration}}{\text{force per unit mass due to perturbation pressure gradient}} = \left(\frac{H}{L}\right)^2 Ro$$

which is identical to the result we obtained for the non-rotating case, except that now the term which must be small in order for the Hydrostatic Approximation to be valid has the Rossby Number in it. In the lab we saw one consequence of this: even for flows with order-one aspect ratio, $H/L = 1$, as long as $Ro << 1$ there was negligible vertical shear of the horizontal flow. Recall that the requirement for the vertical shear to be small in the shallow water equations was that the pressure be hydrostatic. This result is specific to *constant density* fluid flows. If the density field has lateral variation then you can create vertical shear (like in the thermal wind equations), even though the flow is hydrostatic.